temperature and emits long waves. These are strongly absorbed according to the prevailing physical conditions, as the relative amounts of dry air and aqueous vapor, coefficients of absorption for different wave lengths, and so on. Convection currents also enter into the result, and, indeed, the complex function here displayed requires much careful examination before further conclusions can be stated. The temperatures diminish generally with the height after leaving the 400-meter level, but the diurnal period gives a maximum of cold at midday and a minimum about midnight. In the summer months, on the other hand—June, July, August—the inverted temperature distribution does not exist relatively to the surface, but the curves are all of the same general type, with a maximum temperature in the late afternoon, and minimum in the early morning. In the transition months—April, May, September, October, November—the diurnal temperature curve has two maxima, 8 a. m. and 8 p. m., and two minima, 2 a. m. and 2 p. m. The process of transition can be followed in the several levels from month to month, and it is a very interesting phenomenon.

The second important feature of these curves is the building of a semidiurnal period in the temperature at the elevation 400 to 600 meters, in all months in the year, with the maxima at 8 to 9 a. m. and 9 to 10 p. m. They are seen very distinctly represented in May and September, where they are formed up to the very top of the diurnal disturbance. The single diurnal period at the surface is replaced by a double diurnal wave at 400 meters, and this appears quite plainly in every month except July, where it probably is nearly extinct. In the higher levels, above 800 meters, there is a tendency for the double periods to contract the maxima from the 9 a. m., 9 p. m. hours nearer toward midday, and form two crests or a single crest near midday, especially in the winter months. It will be shown in the next paper of this series that those superposed temperature waves, having their maxima disposed as just explained, are competent to produce the diurnal variation of the barometric pressure in the single, double, and triple components, into which the observed pressure at the surface is usually resolved by the Fourier Series of Harmonics. Mr. Clayton has obtained similar curves of temperature at 500, 1000, 1500 meters, as shown on fig. 5 of his paper on "The diurnal and annual periods of temperature."—Annual Harvard College Observatory, Vol. LVIII, Part I, 1904, though they are composites of the several curves really belonging to different months of the year. It will be seen from our curves that mean annual values computed from observations taken in all parts of the year are correct only for certain limited intervals, in which the varying temperatures pass through such special values. Similarly, all discussion of data depending upon mean values made up in this way can have only a limited application in deducing daily free air temperatures throughout the year. This disclosure of the fact that the temperature curves differ according to the elevation from the one observed at the surface opens up the possibility of explaning not only the semidiurnal and triple diurnal barometric waves, but also the movements of the ions in the atmosphere in their relation to the electric potential gradient, the coefficient of neutralization and number of ions, in the connection with other meteorological phenomena, and the variations of the diurnal magnetic field in all latitudes of the earth. These researches will be explained in the other papers of the series.

MATHEMATICAL THEORY OF ICE FORMATION.

By S. TETSU TAMURA.

It is well known to the mathematical physicist that the first to give the most complete mathematical theory of heat conduction was Joseph Fourier, who belonged to the constellation of the most brilliant men of science in the time of Napoleon Bonaparte. The entire subject of heat conduction was created by Fourier's "Théorie Analytique de la Chaleur", and all its later experimental developments have been suggested, or rendered possible, by this immortal work. Side by side with La Place's and Poisson's Equations, Green's Theorem, and Lagrange's Equation, Fourier's Theory, or the analysis known to mathematicians as Fourier's Method, has played a wonderful rôle in the whole domain of mathematical and experimental physics. Lord Kelvin once said that Fourier's work is one of the greatest contributions to the nineteenth century. "Fourier's exquisitely original methods," said the late Professor Tait, "have been the source of inspiration of some of the greatest mathematicians, and the mere application of one of its simplest portions to the conduction of electricity has made the name of Ohm famous".

Among those problems which afford us examples of the application of Fourier's beautiful analysis, there are two groups of the geophysical questions which are extremely important, as well as absorbingly interesting. To the first group belong the problem of the distribution of the earth's internal heat and its consequent effect upon the temperature near the earth's surface, and the problem of the penetration of the sun's heat into the crust of the earth. These problems of widely general and of peculiar mathematical interest were first attacked by Fourier himself and Poisson, the greatest contemporary of Lagrange, and La Place. Some forty years ago, Lord Kelvin startled geologists by telling them that Fourier's Theory forbids such long intervals of time as they were in the habit of assigning to the aggregate of paleontological phenomena. The problems of the second group are rather of smaller feature, but none the less beautiful and interesting as the applications of Fourier's analysis. They are problems of ice formation, especially in the polar sea, and of the penetration of frosts into the ground. These questions, with whose discussion we are now concerned, seem to have been attacked by very few scientists, notably by Franz Neumann who was an eminent mathematical physicist of his day, and Julius Stefan, the famous physicist. Neumann's analysis[1] appears in Riemann-Weber's Partielle Differentialgleichungen der Mathematischen Physik, Bd. II, § 49, and Stefan's memoir[2], "Ueber die Theorie der Eisbildung, insbesondere über die Eisbildung in Polarmeere" in Wied. Ann. Bd. XLII.

Suppose that the surface of a mass of water is in contact with another surface whose temperature is θ_s. θ_s may be either constant or variable; but it must be always below freezing. Under this surface there will be formed a layer of ice, whose thickness, ε, is a function of the time, t. The temperature, θ, of the mass of ice is itself a function of the time, t, and of the distance, x, from the surface.

Let

θ_1 = temperature in ice.
θ_2 = temperature in water.

$$\left.\begin{array}{l} k_1 = \text{conductivity of ice.} \\ k_2 = \text{conductivity of water.} \\ \rho_1 = \text{density of ice.} \\ \rho_2 = \text{density of water.} \\ c_1 = \text{specific heat of ice.} \\ c_2 = \text{specific heat of water.} \\ a_1^2 = \dfrac{k_1}{c_1\rho_1} = \text{diffusivity of ice.} \\ a_2^2 = \dfrac{k_2}{c_2\rho_2} = \text{diffusivity of water.} \end{array}\right\} \text{Assumed as constant.}$$

Then the following equations hold:

$$\begin{aligned} \frac{\partial\theta_1}{\partial t} &= a_1^2\frac{\partial^2\theta_1}{\partial x^2} \text{ in ice, or for } 0 < x < \varepsilon. \\ \frac{\partial\theta_2}{\partial t} &= a_2^2\frac{\partial^2\theta_2}{\partial x^2} \text{ in water, or for } \varepsilon < x. \end{aligned} \tag{1}$$

[1] Also in Wien. Ahad. Sitz. ber. Bd. 98, Abth, II a.
[2] The title, "Vordringen des Frosts."

The temperature of the boundary surface of ice and water (at $x=\varepsilon$) will be always equal to 0° C, and there will be new ice formed continually. If ε is added by $d\varepsilon$ in the time dt, the quantity of heat set free thereby is

$$\lambda\, \rho_1\, d\varepsilon$$

where λ represents the latent heat of fusion of ice and ρ_1 its density. The quantity of heat that flows outward through the unit area of the lowest sheet of ice is given by

$$-k_1\left(\frac{\partial\theta_1}{\partial x}\right)_{x=\varepsilon}dt$$

and at the same time the quantity of heat that flows to the boundary surface upward from the water is

$$k_2\left(\frac{\partial\theta_2}{\partial x}\right)_{x=\varepsilon}dt$$

Hence we obtain the following double surface condition:

$$\left(k_1\frac{\partial\theta_1}{\partial x}-k_2\frac{\partial\theta_2}{\partial x}\right)_{x=\varepsilon}=\lambda\rho_1\frac{\partial\varepsilon}{\partial t} \tag{2}$$

and the other conditions are

$$\theta_1=\theta_s \quad \text{for } x=0 \tag{3}$$

$$\theta_1=\theta_2=0 \quad \text{for } x=\varepsilon \tag{4}$$

also

$$\theta_2=\theta_1 \quad \text{for } x=\delta \tag{5}$$

where

$$\delta>\varepsilon$$

Lastly, if $t=0$, θ_1 for $0<x<\varepsilon$ and θ_2 for $\varepsilon<x<\delta$ may be considered as functions of x.

Now our task will be to solve the two principal equations (1), subjected to the conditions (2), (3), (4), (5); but their general solution has not yet been worked out. The condition (2) contains some unknown function ε, which is not linear, and we have not been able to build up a general solution by summing up a number of particular solutions. Franz Neumann, however, gave in his lectures[3] at Königsberg a solution of our equations under the conditions (2), (3), (4) and

$$\theta_2=\theta_1 \text{ for } x=\infty \tag{6}$$

instead of (5).

To begin with, let us work out the general solution of the equation for heat condition,

$$\frac{\partial\theta}{\partial t}=a^2\frac{\partial^2\theta}{\partial x^2} \tag{7}$$

under the conditions

$$\theta=0 \text{ for } x=0 \tag{8}$$

$$\theta=f(x) \text{ for } t=0. \tag{9}$$

It is easily seen that

$$\theta=e^{-\alpha^2 a^2 t}\sin \alpha x \qquad [\alpha \text{ being wholly unrestricted}]$$

satisfies the equation (7) and the conditions (8) and (9). Therefore it is one of our particular solutions. From this we can build up by Fouriers' theorem[4] the following expression:—

$$f(x)=\frac{2}{\pi}\int_0^\infty d\alpha\int_0^\infty f(\lambda)\sin\alpha x\sin\alpha\lambda\, d\lambda$$

and therefore we have

$$\theta=\frac{2}{\pi}\int_0^\infty d\alpha\int_0^\infty e^{-\alpha^2a^2t}f(\lambda)\sin\alpha x\sin\alpha\lambda\, d\lambda \tag{10}$$

as our required solution; for $\theta=0$, when $x=0$

$$\theta=f(x)=\frac{2}{\pi}\int_0^\infty d\alpha\int_0^\infty f(\lambda)\sin\alpha x\sin\alpha\lambda\, d\lambda, \text{ when } t=0.$$

Now[5]

$$\int_0^\infty e^{-a^2x^2}\cos bx\,dx=\frac{\sqrt{\pi}}{2a}e^{-\frac{b^2}{4a^2}}.$$

Therefore (10) becomes

$$\theta=\frac{1}{2a\sqrt{\pi t}}\int_0^\infty f(\lambda)\left(e^{-\frac{(\lambda-x)^2}{4a^2t}}-e^{-\frac{(\lambda+x)^2}{4a^2t}}\right)d\lambda \tag{11}$$

or, if we put

$$\beta=\frac{\lambda-x}{2a\sqrt{t}} \text{ or } \lambda=x+2a\sqrt{t}\,\beta,$$

(11) may be reduced to the form

$$\theta=\frac{1}{\sqrt{\pi}}\left[\int_{-\frac{x}{2a\sqrt{t}}}^\infty e^{-\beta^2}f(x+2a\sqrt{t}\beta)d\beta-\int_{\frac{x}{2a\sqrt{t}}}^\infty e^{-\beta^2}f(-x+2a\sqrt{t}\beta)d\beta\right] \tag{12}$$

[6]

If the initial temperature be constant and equal to θ_0, or

$$\theta=f(x)=\theta_0 \text{ for } t=0, \tag{13}$$

the expression (12) is reduced to

$$\theta=\frac{\theta_0}{\pi}\left[\int_{\frac{x}{2a\sqrt{t}}}^\infty e^{-\beta^2}d\beta-\int_{-\frac{x}{2a\sqrt{t}}}^\infty e^{-\beta^2}d\beta\right]$$

$$=\frac{2\theta_0}{\pi}\int_0^{\frac{x}{2a\sqrt{t}}}e^{-\beta^2}d\beta. \tag{14}$$

If we suppose that

$$\theta(x)=\frac{2}{\sqrt{\pi}}\int_0^x e^{-\beta^2}d\beta, \tag{15}$$

[7]

then for $x=0$,

$\theta(0)=0$,

for $x=\infty$,

$$\theta(\infty)=\frac{2}{\sqrt{\pi}}\int_0^\infty e^{-\beta^2}d\beta=\frac{2}{\sqrt{\pi}}\times\frac{\sqrt{\pi}}{2}=1,$$

and for the negative value of x,

$$\theta(-x)=\frac{2}{\sqrt{\pi}}\int_0^{-x}e^{-\beta^2}d\beta=-\frac{2}{\sqrt{\pi}}\int_0^x e^{-\beta^2}d\beta$$

whence

$\theta(-x)=-\theta(x)$.

As

$$e^{-\beta^2}=1-\frac{\beta^2}{1!}+\frac{\beta^4}{4!}-\frac{\beta^6}{6!}+\ldots.$$

$$\theta(x)=\frac{2}{\sqrt{\pi}}\int_0^x e^{-\beta^2}d\beta$$

$$=\frac{2}{\pi}\left(\beta-\frac{\beta^3}{3.1!}+\frac{\beta^5}{5.2!}\cdots\cdots\right)$$

$$=\frac{2}{\pi}\sum_{\nu=0}^{\nu=\infty}\frac{(-1)^\nu x^{2\nu+1}}{\nu!\,(2\nu+1)}.$$

Therefore

$$\theta=\frac{2\theta_0}{\pi}\int_0^{\frac{x}{2a\sqrt{t}}}e^{-\beta^2}d\beta$$

$$=\theta_0\,\theta\left(\frac{x}{2a\sqrt{t}}\right) \tag{16}$$

or when expanded,

$$\theta=\frac{2\theta_0}{\pi}\left(\frac{x}{2a\sqrt{t}}-\frac{x^3}{3(2a\sqrt{t})^3}+\frac{x^5}{5.2!(2a\sqrt{t})^5}\cdots\right)$$

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[3] Riemann—Weber, Partielle Differentialgleichungen; Bd. II, § 49.
[4] Riemann—Weber, Bd. I, § 17. Byerly's Fourier's Series, Art. 32.
[5] Byerly's Int. Cal., Art. 94 (2).
[6] Byerly's Fourier's Series, Art. 50. Weber-Riemann, Bd. II, pp. 36 and 37.
[7] Weber-Riemann, Bd. I, §. 26.

$$= \frac{2\theta_0}{\pi} \sum_{\nu=0}^{\nu=\infty} \frac{(-1)^\nu \left(\frac{x}{2a\sqrt{t}}\right)^{\nu+1}}{\nu!(2\nu+1)}.$$

Thus we can easily see that $\theta\left(\frac{x}{2a\sqrt{t}}\right)$ satisfies our principal equations (1). If A_1, B_1, A_2, B_2, are constant,

$$\left.\begin{aligned} \theta_1 &= A_1 + B_1\theta\left(\frac{x}{2a_1\sqrt{t}}\right) \\ \theta_2 &= A_2 + B_2\theta\left(\frac{x}{2a_2\sqrt{t}}\right) \end{aligned}\right\} \quad (17)$$

must also satisfy our equations.

$$\theta\left(\frac{x}{2a\sqrt{t}}\right) = 0 \text{ for } x = 0$$

$$\theta\left(\frac{x}{2a\sqrt{t}}\right) = 1 \text{ for } x = \infty$$

$\theta\left(\frac{x}{2a\sqrt{t}}\right)$ becomes constant also, if x is proportional to $\sqrt{t}$ or if $\varepsilon = b\sqrt{t}$ when $x = \varepsilon$. (18)

In order that the solutions (17) satisfy (2), (3), (4), and (6), we must have

$$\left.\begin{aligned} A_1 &= \theta_s \\ A_1 + B_1\theta\left(\frac{b}{2a_1}\right) &= 0 \\ A_2 + B_2\theta\left(\frac{b}{2a_2}\right) &= 0 \\ A_2 + B_2 &= \theta_i. \end{aligned}\right\} \quad (19)$$

$$\frac{k_1 B_1}{a_1\sqrt{\pi t}} e^{-\frac{b^2}{4a_1^2}} - \frac{k_2 B_2}{a_2\sqrt{\pi t}} e^{-\frac{b^2}{4a_2^2}} = \frac{\lambda\rho_1}{2\sqrt{t}} b.$$

From (19) we have

$$B_1 = \frac{\theta_s}{\theta\left(\frac{b}{2a_1}\right)} \qquad B_2 = \frac{\theta_i}{1 - \theta\left(\frac{b}{2a_2}\right)}$$

whence

$$\frac{k_1\theta_s e^{-\frac{b^2}{4a_1^2}}}{a_1\theta\left(\frac{b}{2a_1}\right)} + \frac{k_2\theta_i e^{-\frac{b^2}{4a_2^2}}}{a_2\left[1 - \theta\left(\frac{b}{2a_2}\right)\right]} = -\lambda\rho_1 b\frac{\sqrt{\pi}}{2}. \quad (20)$$

From this last transcendental equation we may be able to find the constant b or $\frac{\varepsilon}{\sqrt{t}}$. Since $\varepsilon = b\sqrt{t}$, it follows that the thickness of ice increases in direct proportion to the square root of the time, and also it follows that the formation of ice is slow with the increase of depth.

If b is found, A_1, A_2, B_1, and B_2, which appear in (17), may be easily found from (19).

If we take θ_s positive and θ_i negative, the formulæ represent the laws of fusion of ice.

It must not, however, be forgotten that the whole analysis, however elegant it may be, has been based upon many objectionable assumptions. In the first place, the change of the density of water when frozen into ice was entirely neglected in the analysis. The assumptions

$$\theta_1 = \theta_s \text{ (constant) for } x = 0$$
$$\theta_2 = \theta_i \text{ (constant) for } x = \infty$$

are another objectionable feature. Lastly, it must be remembered that the solutions are merely particular solutions of our equations.

Julius Stefan simplified the problem by the assumption that the temperature of water below the boundary surface of ice and water is everywhere and always equal to zero. Our equations and boundary conditions (1), (2), (3), (4), (5) then become

$$\frac{\partial\theta_1}{\partial t} = a_1^2\frac{\partial^2\theta_1}{\partial x^2} \text{ for } 0 < x < \varepsilon. \quad (21)$$

$$k_1\left[\frac{\partial\theta_1}{\partial x}\right]_{x=\varepsilon} = \lambda\rho_1\frac{d\varepsilon}{dt} \quad (22)$$

$$\theta_1 = -\theta_s \text{ for } x = 0 \quad (23)$$

$$\theta_1 = 0 \text{ for } x = \varepsilon \quad (24)$$

As θ_1 is a function of the time and place,

$$d\theta_1 = \frac{\partial\theta_1}{\partial x}dx + \frac{\partial\theta_1}{\partial t}dt$$

when $x = \varepsilon$, $\theta = 0$, whence

$$\frac{\partial\theta_1}{\partial t} + \left(\frac{\partial\theta_1}{\partial x}\right)_{x=\varepsilon}\frac{d\varepsilon}{dt} = 0$$

$$\frac{\partial\theta_1}{\partial t} = -\frac{a_1^2 c_1}{\lambda}\left(\frac{\partial\theta_1}{\partial x}\right)^2_{x=\varepsilon} \quad (25)$$

It is easily seen that if $\xi = \varepsilon - x$, then the expression[8]

$$-\frac{c\theta_1}{\lambda} = \frac{1}{2!}\frac{d\,\xi^2}{a_1^2 dt} + \frac{1}{4!}\frac{d^2\xi^4}{a_1^4 dt^2} + \ldots\ldots \quad (26)$$

satisfies (21) and the condition $\theta_1 = 0$ for $x = \varepsilon$.

Let us differentiate this expression with respect to x and then we obtain

$$-\frac{c}{x}\frac{\partial\theta_1}{\partial x} = -\frac{1}{1!}\frac{d\xi}{a^2 dt} - \frac{1}{3!}\frac{d^2\xi^2}{a^2 dt^2} \quad \ldots\ldots$$

When $x = \varepsilon$, this satisfies (24).

Since $\theta_1 = -\theta_s$ for $x = 0$, it follows that

$$\frac{c\theta_s}{\lambda} = \frac{1}{2!}\frac{d\varepsilon^2}{a^2 dt} + \frac{1}{4!}\frac{d^2\varepsilon^4}{a^4 dt^2} + \ldots\ldots \quad (27)$$

If the thickness of the layer of ice ε is given as a function of time t, then θ_s may be easily determined; but, if θ_s is given, it is in general difficult to determine ε.

Now, if we suppose that the temperature θ_s of the upper surface is constant, the right side of the expression (27) must be constant. This condition is fulfilled, if the thickness of ice increases in proportion to the square root of the time, or

$$\varepsilon^2 = 2p^2a^2t \quad (28)$$

where p is constant. Then from (27) we have

$$\frac{c\theta_s}{\lambda} = \frac{p^2}{1} + \frac{p^4}{1\cdot3} + \frac{p^6}{1\cdot3\cdot5} + \ldots\ldots \quad (29)$$

For the first approximation

$$\frac{c\theta_s}{\lambda} = p^2 = \frac{\varepsilon^2}{2a^2t} \quad (30)$$

whence

$$\varepsilon^2 = \frac{2ca^2\theta_s t}{\lambda}. \quad (31)$$

For the second approximation

$$\frac{c\theta_s}{\lambda} = p^2 + \frac{p^4}{3} \text{ or } \frac{1}{3}p^4 + p^2 - \frac{c\theta_s}{\lambda} = 0$$

$$p^2 = \frac{-1 \pm\sqrt{1 + \frac{4}{3}\frac{c\theta_s}{\lambda}}}{\frac{2}{3}} = +\frac{3}{2}\left(\sqrt{1 + \frac{4c\theta_s}{3\lambda}} - 1\right)$$

$$\varepsilon^2 = 3a^2t\left(\sqrt{1 + \frac{4c\theta_s}{3\lambda}} - 1\right). \quad (32)$$

The above formulæ may be obtained by the following method:

The expression

[8] Stefan's "Ueber die Theorie der Eisbildung." Wied Ann. Bd. XVII

$$\theta_1 = A\int_{\frac{x}{2a\sqrt{t}}}^{\frac{p}{\sqrt{2}}} e^{-\beta^2} d\beta \tag{33}$$

in which A and p are constants, satisfies the equation (21).

If $x = 0$, θ_1 will take a constant value.

$$\theta_s = A\int_0^{\frac{p}{\sqrt{2}}} e^{-\beta^2} d\beta, \tag{34}$$

which is the expression for the temperature of the upper surface.

From (33)

$$\theta_1 = 0, \text{ when } \frac{x}{2a\sqrt{t}} = \frac{p}{\sqrt{2}},$$

and by (24) $\theta_1 = 0$, when $x = \varepsilon$,

whence $$\frac{\varepsilon}{2a\sqrt{t}} = \frac{p}{\sqrt{2}},$$

or, as expressed in (28), $\varepsilon^2 = 2p^2 x 2t.$ (28)

By (25) we can determine the value of p. Let us differentiate θ_1 in (33) with respect to x and t. Then we have

$$\frac{\partial\theta_1}{\partial t} = A\, e^{-\frac{x^2}{4a^2t}} \frac{x}{2at\sqrt{t}}.$$

$$\frac{\partial\theta_1}{\partial x} = -A\, e^{-\frac{x^2}{4a^2t}} \frac{1}{2a\sqrt{t}}.$$

Introducing these expressions into (25), we obtain

$$A\, e^{-\frac{p^2}{2}} \frac{p}{2\sqrt{2}\,t} = \frac{a^2c}{\lambda} A^2 e^{-p^2} \frac{1}{4a^2t}.$$

With (34)

$$\frac{p}{\sqrt{2}} e^{\frac{p^2}{2}} \int_0^{\frac{p}{\sqrt{2}}} e^{-\beta^2} d\beta = \frac{c\theta_s}{2\lambda}, \tag{35}$$

which serves for the determination of p. By expanding (35), we reach the formula (29)

$$\frac{c\theta_s}{\lambda} = \frac{p^2}{1} + \frac{p^4}{1\cdot 3} + \frac{p^6}{1\cdot 3\cdot 5} + \dots\dots \tag{29}$$

and, as before shown,

$$p^2 = \frac{c\theta_s}{\lambda} \text{ or } p = \sqrt{\frac{c\theta_s}{\lambda}}, \tag{30}$$

$$\varepsilon^2 = \frac{2ca^2\theta_s t}{\lambda}, \tag{31}$$

etc., etc.

All these formula hold only when θ_s is constant.

Another particular solution which may be made to satisfy (21) and (25) is

$$\theta_1 = \frac{A}{\alpha}\left(e^{\alpha t - qx} - 1\right) \tag{36}$$

where A, α, and q are constants. Substituting this expression for θ_1 in (21), we find that $\alpha = a^2 q^2$.

Now

$$\theta_1 = 0, \text{ for } \alpha t - qx = 0$$
$$\theta_1 = 0, \text{ for } x = e$$

whence

$$\alpha t - q\varepsilon = 0$$

$$\varepsilon = \frac{\alpha t}{q} = q\, a^2 t. \tag{37}$$

The expression shows that the thickness of ice may increase in direct proportion with the time.

From (25)

$$A = \frac{c\, a^2 q^2 A^2}{\alpha^2 \lambda} = \frac{cA^2}{\alpha\lambda}.$$

Hence between A and α there exists the following relation:

$$\alpha = \frac{Ac}{\lambda}. \tag{38}$$

For $x=0$, we obtain from (36)

$$\begin{aligned}\theta_s &= \frac{A}{\alpha}\left(e^{\alpha t} - 1\right)\\ &= \frac{A}{\alpha}\left(\frac{\alpha t}{1!} + \frac{\alpha^2 t^2}{2!} + \frac{\alpha^3 t^3}{3!} + \dots\dots\right)\\ &= At + \frac{c}{\lambda}\frac{A^2t^2}{2!} + \frac{c^2}{\lambda^2}\frac{A^3t^3}{3!} + \dots\dots\end{aligned} \tag{39}$$

Each of these formulas, however, contains some unknown constants α or A.

If we introduce the expression (37) into (27), we reach another expression,

$$\frac{c\,\theta_s}{\lambda} = q^2 t + \frac{q^4 t^2}{2!} + \frac{q^6 t^3}{3!} + \dots\dots = e^{q^2 t} - 1. \tag{40}$$

If ε is very small, the formula (27) may be reduced to the form,

$$\frac{c\,\theta_s}{\lambda} = \frac{2}{2a^2}\frac{d\varepsilon^2}{dt}$$

whence

$$\varepsilon^2 = \frac{2k}{\lambda\rho_1}\int_0^t \theta_s dt \tag{41}$$

θ_s may be variable, and if it is a function of t,

$$\varepsilon^2 = \frac{2k}{\lambda\rho}\int_0^t f(t)\, dt.$$

If we assume that the temperature of ice increases uniformly from its upper surface downward, the quantity of heat $\frac{k\theta_s}{\varepsilon} dt$ flows upward through the ice in the time dt. In the same time, a layer of ice whose thickness is $d\varepsilon$ is formed and the quantity of heat $\lambda\rho_1 d\varepsilon$ is set free; therefore we have

$$\frac{k\,\theta_s}{\varepsilon} dt = \lambda\rho_1 d\varepsilon$$

from which

$$\varepsilon^2 = \frac{2k}{\lambda\rho_1}\int_0^t \theta_s dt$$

which was already obtained in (41).

If θ_s is constant,

$$\varepsilon = \sqrt{\frac{2\,\theta_s k t}{\lambda\rho_1}} \tag{42}$$

It is to be regretted that the analysis, so far, is very incomplete and unsatisfactory. In fact, some formulas have been deduced upon the assumption that the temperature of the upper surface of ice θ_s is constant, and others upon the assumption that the thickness of ice ε is very small or that the temperature gradient in ice is constant. Moreover, the whole operation is based upon the assumptions that the temperature of water is everywhere and always equal to zero, and that the heat flows upward only, and the latent heat of fusion of ice never warms up the water next the ice. The change in density, when water freezes into ice, was also neglected.

If it be satisfactorily completed, however, the theory of ice formation may be applicable to a few interesting problems. When a lake or the polar seas are covered with a thin uniform layer of ice, the ice grows gradually thicker if the temperature of the air be below the freezing point. This is caused by the passage of heat through ice from the water immediately below it. Thus the layer of water next the ice freezes, setting free the latent heat of fusion of ice. As the temperature is the same throughout any horizontal layer, the transference of heat is vertically upward, or perpendicular to such layers. Stefan applied some of his formulas to the phenomena of ice formation in the polar seas, notably in the Gulf of Boothia, Assistance Bay, Port Bowen, Walker Bay, Camden Bay, and others, and computed the value of a_1^2, which is $\frac{k_1}{\rho_1 c_1}$, to be 0,0042 (C. G. S.) which value lies between the value of Neumann (0.0057) and that of Forbes (0.00223). This is rather a remarkable result. But Stefan's application must be criticized because he took simply for the surface condition $\theta=\theta_s$ for $x=0$. θ_s is primarily the temperature of the outer surface of ice, and not that of the atmosphere itself, as Stefan assumed. Hence our surface condition must be $\left[k_1 \frac{\partial \theta_1}{\partial x}\right]_{\lambda=0} = \sigma(\theta_1-\theta_s)$ when σ is the emissivity of ice and θ_s may be the temperature of the atmosphere, if we assume that heat is radiated from ice surface toward the atmosphere only and not toward space. The fact is, however, that the influence of the temperature of the atmosphere on ice formation is seemingly small. The natives of Bengal, in India, make ice in fields freely exposed to the sky. Small excavations are made in the black loam soil, at the bottom of which are spread small sheaves of rice straw, and upon the top of this is placed light, loose straw to the depth of one and one-half feet. Upon this shallow and porous earthen dishes filled with water are exposed during clear nights. Ice is produced in large quantities when the air temperature is 16° to 20° F. above the freezing point. This shows that ice formation is in great part due to radiation to space. Hence θ_s must be the so-called sky temperature, which represents both the siderial and the atmospheric temperature.

If the temperature of the air is higher than that of ice, the ice begins to melt gradually, and at the same time a part of the heat flows into the ice downward. This is equivalent to saying that cold flows outward from the interior of ice and retards the melting of the surface of ice by the warmer air. Therefore, the principles of ice formation may be applicable to this problem, if we give the formulas the opposite signs.

The penetration of frost into the moist soil is another interesting problem in the application of the theory of ice formation. If the temperature of the earth's surface is cooled below the freezing point, the frost is formed over the earth's surface and penetrates gradually deeper into the earth, which process is exactly similar to the formation of ice in a lake or in the polar seas.

All these problems are examples of heat conduction in one dimension, which is the simplest of all cases. If we suppose that a vessel filled with water is surrounded by brine, the layer of water nearest to the brine begins to freeze. Heat flows outward from the water and the layer of ice will grow toward the center of the vessel. If the vessel be a parallelopiped, our subject becomes the difficult problem of heat conduction in a parallelopiped of ice. If the vessel be a cylinder the problem becomes the most difficult, and may possibly require the assistance of the theory of Bessel's functions, if, indeed, the problem be soluble at all. It is in general very difficult to determine the variation of temperature in a limited body.

Prof. Cleveland Abbe asked me sometime ago if I could solve the problem of ice formation more completely, and if I would also prepare a paper to point out to American meteorologists how much on the subject has been done by mathematical and experimental physicists, and what ought to be done by meteorologists. Now I communicate this preliminary paper to the WEATHER REVIEW, with the hope that the problem may interest some of the theoretical as well as practical meteorologists, and also with the expectation that I may offer in the near future some more general solution of the problem than those above given.

NOTES AND EXTRACTS.

THE FOURTH INTERNATIONAL CONFERENCE ON AERIAL RESEARCH.

In the Meteorologische Zeitschrift for January Dr. A. De Quervain gives a general report on the proceedings of the fourth conference of the international committee for scientific ballooning or aerial investigation. The conference was held at St. Petersburg, August 29–September 3, 1904, in the rooms of the Imperial Academy of Sciences.

After enumerating the papers read at the conference, De Quervain gives the following abstract of Professor Hergesell's report on his kite work on the Atlantic Ocean. This work, which was carried out on the yacht belonging to the Prince of Monaco, began in the middle of July off the coast of Portugal, and was extended southward to twenty nautical miles southwest of the Canaries. From the middle of August onward the work was prolonged in the direction of the Azores after Hergesell had left the yacht. The trade wind was blowing off the coast of Portugal, and increased in strength in proportion as the yacht moved southward. North of the Canaries there was a northeast wind of seven or eight meters per second. The trade wind diminished with altitude above sea level until it was inappreciable. The kites attained altitudes as high as 4500 meters. The following conditions were observed in the trade region: in the lowest strata of a few hundred meters thick the temperature diminished adiabatically; then followed a sharp passage into a layer having an inverse gradient, generally of the considerable thickness of a thousand meters, in which the temperature rose, and which was generally very dry, namely, relative humidity of 10 or 12 per cent. Above this came another layer, with adiabatic temperature gradients, whose upper limit was not attainable, but which certainly extended to an altitude of 5000 meters. As the kite ascended, the wind backed from northeast to northwest and diminished to a very feeble movement in the inversion layer. A southwest wind, or anti-trade proper, was not observed in these kite ascensions up to 4500 meters. Professor Hergesell considers the westerly winds observed on the peak of Teneriffe as being of a local nature. As to the presence of local winds, very interesting observations were made in the neighborhood of the Canary Islands. A general account of the results of this work will probably soon be published in the Comptes-Rendus, after presentation by the Prince of Monaco to the Academy of Sciences in Paris. The publication in extenso will be made in Hergesell's Beiträgen to the physics of the free atmosphere.

The new edition of the International Cloud Atlas has now been provided for financially and the publication will be hastened.

As to the organization of the international balloon work, the committee decided that not only should the monthly ascensions, on specified dates, continue as heretofore, but also that ascensions should be made as frequently as possible on three consecutive days during two months of 1905, namely, April and also the 29th, 30th, and 31st of August. These last dates were chosen with reference to the great total eclipse of the sun, which will occur on August 30, and on which occasion the Spanish Government desires that ascensions shall be made in Spain within the path of totality. The hour of the day for simultaneous international ascensions was postponed, since the newer meteorographs are so well protected against sunshine that they give correct records of the temperature of the air, and insolation is not so much to be feared. The committee